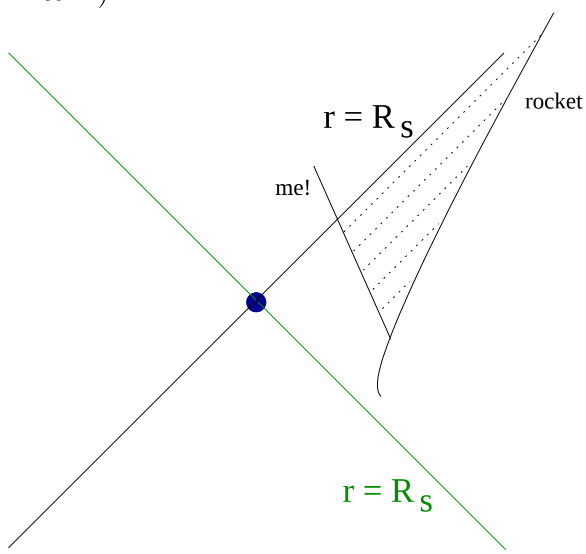
Solutions to Homework Assignment #11 – PHY312

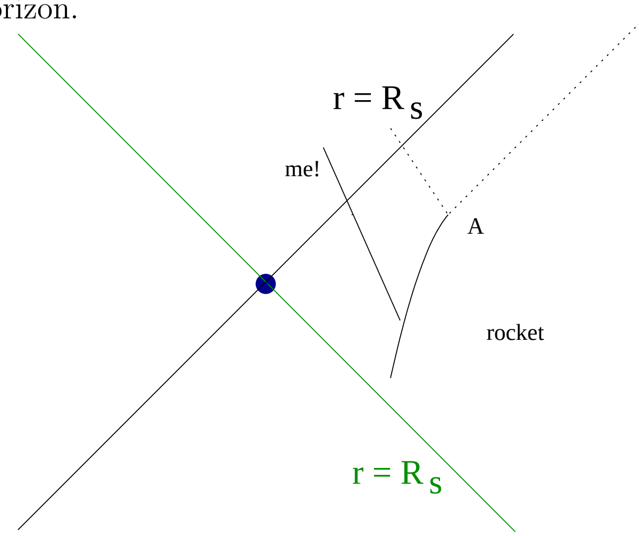
9-4)
A & B)



The diagram looks much like the one we drew for a person falling across an acceleration – because it is essentially the same thing!!! Here the person falls freely, so I have drawn them as a straight line. The light rays are of course at 45 degrees from the vertical, and I have drawn them as being equally spaced along the person's worldline (since they are emitted at regular intervals).

C) Note that the light rays are not received at equal interval, however. Because the rocket accelerates away, it succeeding pulse arrives after a longer and longer interval. Thus, observers in the rocket see the falling person to move and age more and more slowly, and in fact the falling person is never seen to fall across the horizon. Instead, they merely become more and more compressed against the horizon. Of course, they also become more and more dim (since the person only emits a finite amount of energy before crossing the horizon, but it is absorbed over infinite time) and more and more red (in this case, we call the effect gravitational redshift).

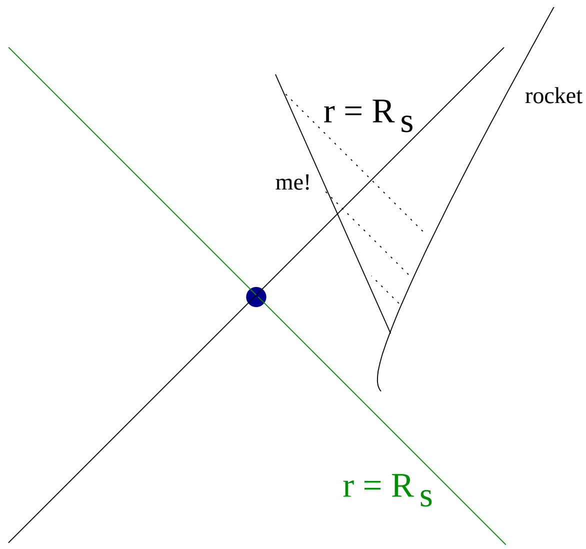
Although they never see you actually cross the horizon, it is impossible for them to wait around for a while and then come and rescue you. This can be seen from the diagram below. The future light cone from event A is shown with dotted lines. Since the rocket must follow a timelike worldline, it is clear that it cannot catch up to the falling person before they cross the horizon.



D) As a freely falling observer, you feel only the gravitational tidal forces. That is, you feel the fact that you are being stretched in the direction away from the black hole (the radial direction) and squished in the direction around the black hole (the angular directions). Two parts of your body separated by a distance L are trying to accelerate toward or away from each other at $c^2 R_s L/(r^3)$. In particular, at the horizon this is $\frac{Lc^2}{R_s^2}$. For a sufficiently big black hole (when r is not too small) this can be quite small and you would not even notice the effect. However, for a smaller black hole the effect will be huge and you will be both crushed and ripped apart. For a solar mass black hole, this relative acceleration is around $5 \times 10^1 0 m/s^2$ for a typical person. However, for the black hole at the center of our galaxy (two million solar masses), it is only $10^{-2} m/s^2$ – an amount too small to notice.

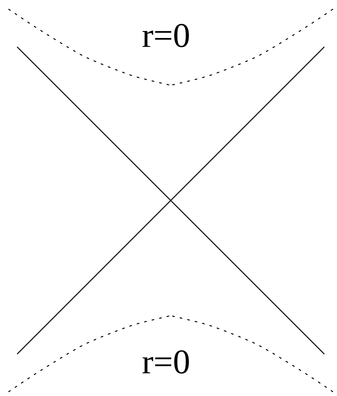
Of course, as you continue to fall the stretching and squishing effect gets larger and larger. For the black hole at the center of our galaxy, the person falling in reaches $r = 0$ (where the tidal forces become inifnite!) roughly ten seconds after falling through the horizon.

E) I have drawn light rays from the rocket to the falling person on the diagram below. The falling person sees the rocket a bit slowed due to the usual time dilation (and more so as the rocket continues to accelerate), but the person sees nothing special at the moment that they fall across the horizon. There is no particular difference between what is seen falling into a large vs. a small black hole.



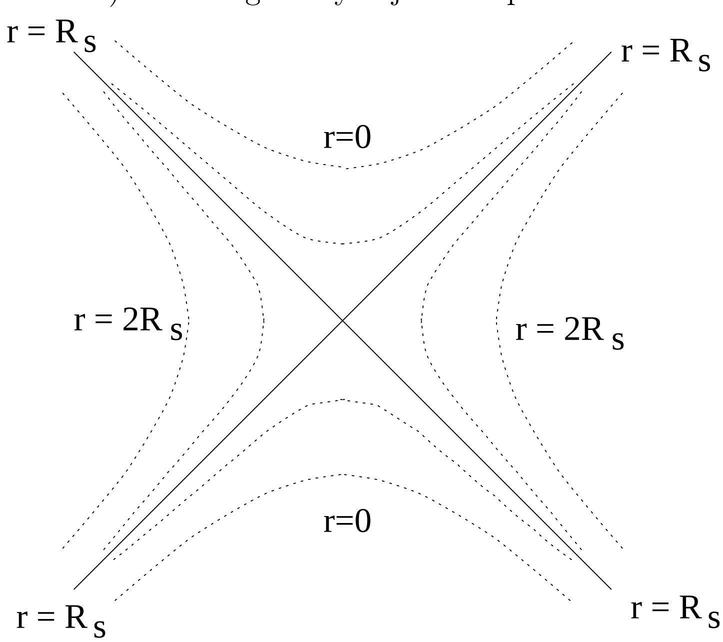
9-6. A) Regions I and III are the 'exterior' regions where you can remain static and go far away from the black hole. Region II is the 'future interior' and region IV is the 'past interior.' The past interior is also sometimes called the 'white hole region' because it is easy to get out of but impossible to get into (without moving faster than light). A 'real' black hole that forms from the collapse of a star or gas cloud has only region II (the future interior) and either region I or III (these are equivalent – it doesn't matter which one you choose).

B) The singularity goes in regions II and IV. The important point is that the singularity is a 'spacelike' line – that is, it is a line which is really a line at some *time.* It is a peculiarity of the Schwarzschild coordinates that the singularity is at '$r = 0$.' As we discussed in class, once you are inside the black hole r is actually a time coordinate, not a distance coordinate (though it does tell you how big circles and spheres 'around' the black hole are). The picture is just what I drew in class:

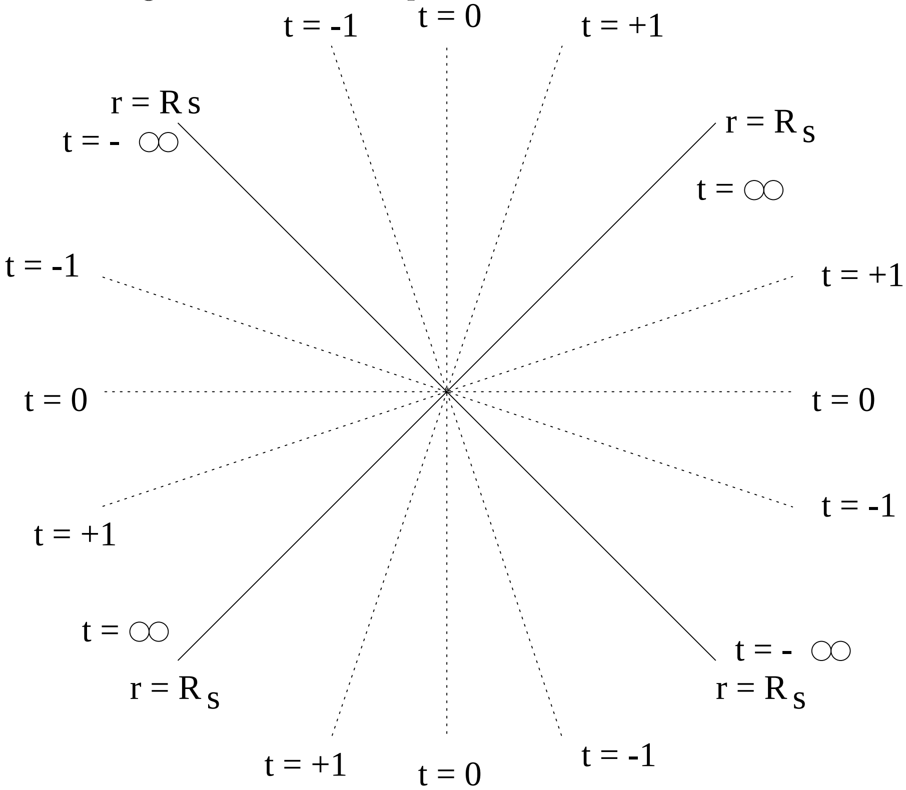


The dotted line above is the singularity.

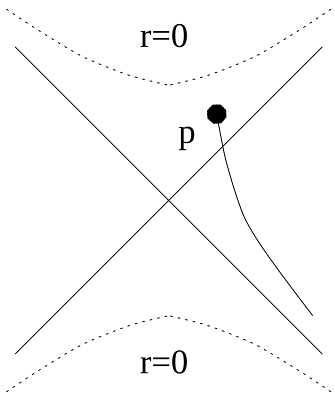
C) The lines of constant r are rather familiar looking hyperbolae (the dotted lines below). The singularity is just the particular such line for which $r = 0$.



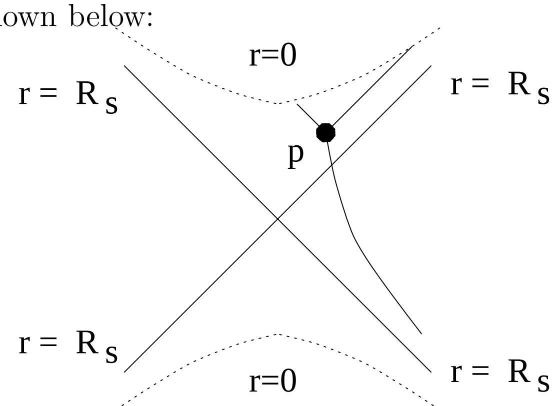
D) The lines of constant t are the dotted lines below. Just as for the family of uniformly accelerated observers near an acceleration horizon in flat spacetime, the lines of constant t all intersect in the middle of the diagram. In regions II and IV, the constant t lines are similar to those in regions I and III, except that here, in the interior, they are *timelike* lines (i.e., they point in a time direction) while they are spacelike lines (pointing in a space direction) outside. Notice that the labeling of the lines is rather interesting with the labeling in region III being 'backwards.' The labels $t = \pm 1$ should not be taken too seriously as I have not given any units with them – they are merely to indicate which lines correspond to positive t and which correspond to negative t. Again, we have already seen this phenomenon with the family of accelerating observers in flat spacetime.



E) If you fall into a black hole from the exterior, your worldline must look something like:



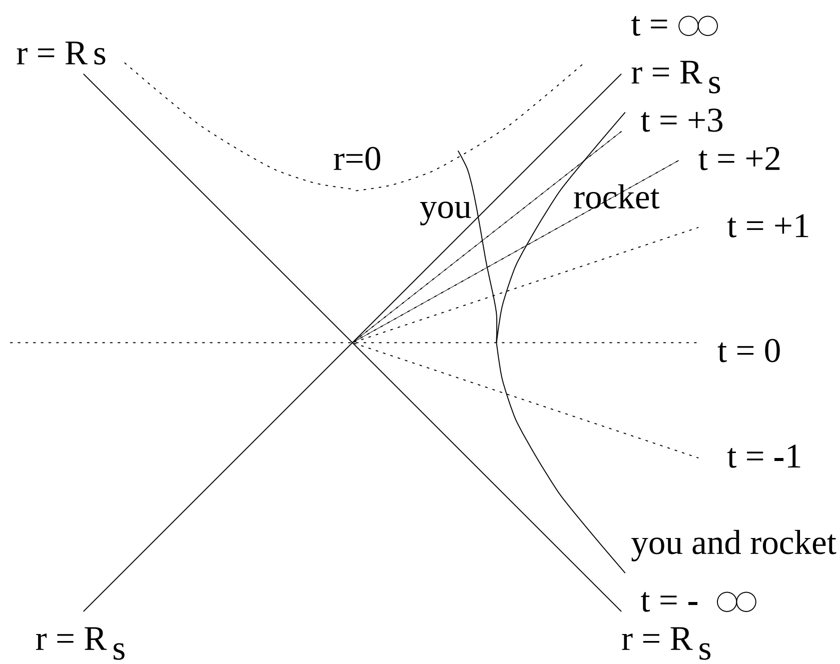
Suppose that you are now at event p. Can you get out? Well, you are only allowed to travel at less than the speed of light. This means that, no matter what you do, you will always stay inside the future light cone of p, shown below:



Clearly, any path that stays inside that light cone will not be able to cross the horizon and, moreover, will hit the singularity.

F) Since the lines $r = R_s$ are the paths of light rays, you can't cross them (in the direction that the light ray is traveling) without going faster than light. So, it is impossible to cross from region I into region IV, or region II into region III, without going faster than light. Thus, from region I, you can only reach region II, not region III.

9-7. A) I'll draw the diagram below and, in direct analogy with the situation for accelerated observers in flat spacetime (problem 5-3), I'll draw in the lines of constant t. It is also useful to draw in the hyperbola $r = 0$ representing the singularity.



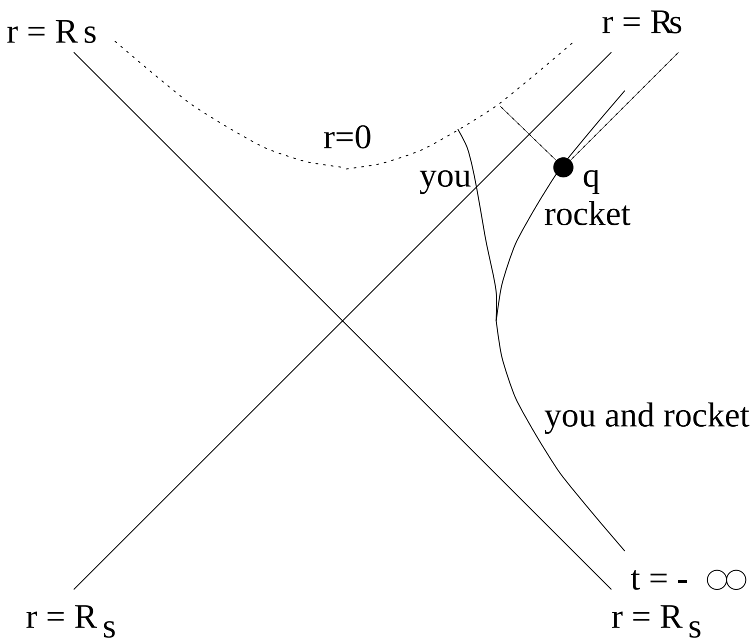
B) Note that, as you fall in, you pass the lines $t = +1$, $t = +2$, $t = +3$ ... passing all of these before you cross the horizon. In fact, before you cross the horizon, you will pass the line $t = const$ for any finite value of t. Now, any measurement of a static observer outside the horizon will correspond to some finite value of t and, as we have just seen, at any finite value of t you are still *outside* the horizon. So, any measurement done by the static observers (or, what is essentially the same, on the rocket ship) will only have to do with events in your life before you cross the horizon.

In particular, when you do cross the horizon, your watch will read some time – let's call it τ. No matter how long static observers outside wait, their measurements will always find that your watch does not yet (quite) read τ. As a result, they find that time for you is running very slowly relative to how it runs for them – in fact, infinitely slowly as you near the horizon. As usual, there will be a corresponding redshift, and you will look more and more red (and then infrared, and then you will be 'visible' only with a microwave or radio antenna ...).

In the above paragraph, I was talking about measurements, but essentially the same thing is true of what the static observers will *see*. Clearly, no light ray from after you cross the horizon can reach them. So, if they watch you through a telescope, no matter how long they wait, they never quite see your watch read τ, and they never quite see you fall across the horizon.

There is another rather interesting effect. Suppose that you are falling toward the horizon head-first. Think about what happens if you draw the worldlines of both your head and your feet. Of course, they both cross the horizon. Now, remember that the static observers will measure proper distance along the $t = const$ lines of simultaneity and that the lines of constant distance from the horizon are just the lines of constant r. You can see that, if the static observers wait for long enough, both your head and feet will cross any line of constant r outside the horizon. Thus, they will find that, if they wait long enough, the distance between your head and the horizon goes to zero *and* that the distance between your feet and the horizon goes to zero. This is true independent of whether you would actually *feel* any discomfort as you fall toward the horizon.

Since the people in the rocket will never actually see (or even measure) you to fall across the horizon no matter how long they wait, it is tempting to ask the question "can they come and rescue you?" Again, we can see the answer from a spacetime diagram. Of course, if they start soon enough after you fall (so that you have not fallen very far), they can certainly rescue you. But what happens if they don't react very fast? So far, I've assumed that the rocket is remaining static and drawn in its worldline. This is represented by the solid line labeled 'rocket' in the diagram below. Suppose that the rocket does not react until point q. The rocket, of course, is constrained to move at less than the speed of light (otherwise, it could equally well move backwards in time, in which case it is already clear that it could rescue you). Since $v < c$, the rocket can only reach the events inside the future light cone of q (dotted lines below). Thus, we see that it cannot reach you before you cross the horizon. In fact, it cannot even reach you before you hit the singularity! The point made above that a *static* observer will always find you outside the horizon holds only for a static observer (or one that is approximately so). A rocket that tries to come down to the horizon to save you is definitely not static, and the conclusion does not apply – it would find that you have already crossed the horizon before the rocket gets there.



C) As described in section 9.4 about 'tidal forces,' what you *feel* is completely determined by the relative accelerations of the freely falling trajectories that would originate from different parts of your body. These are the worldlines that the parts of your body *would* follow if your various tissues didn't hold them together. So, this relative acceleration determines the amount of stress that your tissues must endure in order to hold your body together in its usual shape.

According to the discussion in section 9.4, the relative acceleration of two free fallers (with zero initial velocity) separated by a (small) distance L, at any point outside the black hole is

$$a = \frac{Lc^2 R_s}{r^3}. \quad (1)$$

This is largest at the horizon, where we have

$$a = \frac{Lc^2}{R_s^2}.$$

Now, we have to try some numbers to see how big this is. A human has roughly, $L = 1m$, so we get

$$a = 10^{17} m^3 s^{-2} R_s^{-2}.$$

For a black hole whose mass is roughly that of the sun, $R_s = 3000m$, so $a = 10^{10} m/s^2$!!!. That's a huge number!!! So, you would not in fact survive falling into a black hole with the mass of our sun ... I'm afraid it would be very grisly.

How about a much bigger black hole? Say, one of the smaller ones that might be at the center of a galaxy? They are about 10^6 times as massive as the sun, with a correspondingly greater R_s. So, the acceleration is smaller by a factor of 10^{12}. In other words, $a = 10^{-2} m/s^2$. This is a very *small* number which you could not even feel!!! So, falling across the horizon of a fairly big black hole would not feel like anything at all!! You would feel just like you would if you were falling freely in completely empty space.

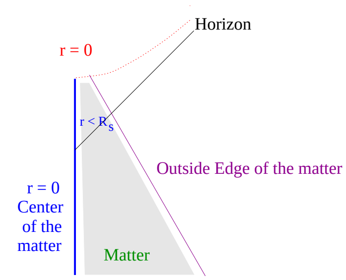
This however, only gets you up to the horizon. When you get inside, your value of r becomes even smaller, and equation (1) tells us that the tidal forces increase. Since we know that you cannot avoid the tidal forces at $r = 0$, these tidal forces grow without bound. At the singularity, the relative acceleration between any two freely falling worldlines becomes infinite. So, your head and feet will want to move away from each other with an infinite acceleration. I'm afraid there's not much that your tissues can do about that. It would not be fun. Just in case you're wondering, if you fell into a 10^6 solar mass Schwarzschild black hole, you would have about a second of proper time before hitting the singularity and getting torn to shreds. (If the black hole was a *really* big one, say 10^{15} solar masses, you would have 10^9 seconds or about 100 years.)

Note that equation (1) does in fact describe tidal forces that attempt to rip your head away from your feet (if you fall toward the black hole feet first). The forces discussed in section 9.3 where equation (1) was derived deal with tidal forces in the 'radial' direction.

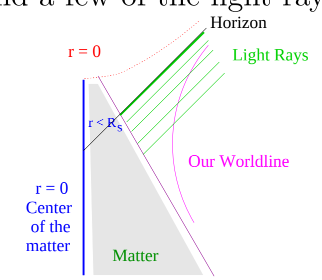
However, there are also tidal forces that act in the direction *around* the black hole. As we discussed, these forces compress you, pressing your right shoulder into your left shoulder. These forces are roughly the same size as the radial forces described by equation (1) and again can be either large or small at the horizon as discussed above. Of course, they become infinite at the singularity.

Now for the question of 'What do you *see* if you watch the rocket as you fall in?' Well, it is clear that you will only see a finite part of the rocket's worldline before you hit the singularity (and expire). Drawing some light rays on our diagram shows that there is a bit of redshift and time dilation (so that you see the rocket slowing down somewhat), but nothing particularly dramatic.

9-8. A) A spacetime diagram of this sort can be found at the end of section 9.3.4. So, all we need to do is to reproduce it here.

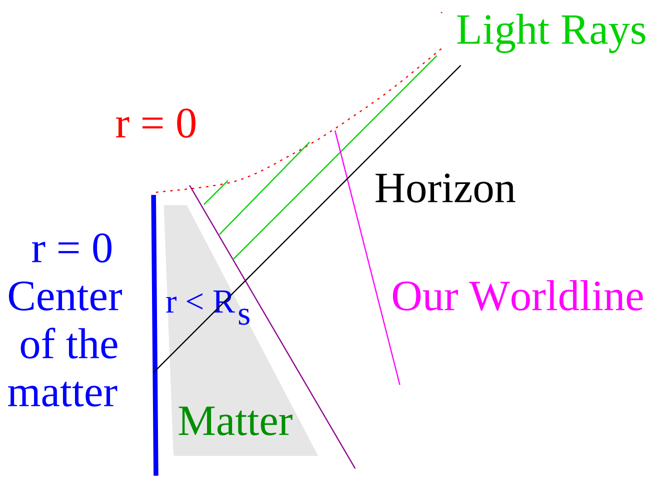


B) Now we just need to use this diagram to figure out what we see if we watch the star collapse. The light that we see comes from the outside edge of the star. Let me draw a new version of this diagram showing our worldline and a few of the light rays coming out from the star.



What we see is that although the light rays are emitted at equal time intervals (according to the star), the proper time interval (along our worldline) between successive rays reaching us becomes larger and larger. The result is that we see the star becoming dimmer and dimmer and redder and redder (since, as usual, this time dilation creates a Doppler effect or 'red-shift'). However, the light ray emitted when the star crosses the horizon never reaches us, so that we never see the star crossing the horizon. The star always appears to be outside of the horizon.

c) Again, the way to solve this problem is just to use our diagram. I'll redraw it one more time showing our new worldline (falling into the black hole!) and some of the light rays emitted by the star after it crosses the horizon:



Here we fall right into the singularity (as in problem 9-4/9-7...), so we get squished and torn to shreds. Oh, well. But let's ignore that for a moment and suppose that we are amazingly strong so that we can survive right up to the very singularity itself. Can we see the star hit the singularity? No. Note that a number of the light rays emitted by the star hit the singularity without reaching us first! There is no way that we could ever see those light rays. This is a general consequence of the fact that the singularity is *spacelike*. No signal can come from one part of the singularity (like, where the star hits the singularity) to another part (like, where we hit the singularity).

9-9. To do this one (the optional problem) you will have had to read section 9.3.5. However, it works just like the discussion in that section. The important point is to realize that, when energy is transported up or down in a gravitational field, it grows or shrinks by the time dilation factor. In the case of the Schwarzschild metric, this is $\sqrt{1 - R_s/r}$.

So, the rock starts out far away with just its 'rest energy' $E = mc^2$. Then, it falls down to the photon sphere, at $r = 3R_s/2$. If the rock fell freely, it's energy at the photon sphere would be $E = mc^2/\sqrt{1 - R_s/r} = mc^2\sqrt{3}$. Note that this is true independent of how big the black hole (or the photon sphere) is, and remember that, for a big black hole, it *is* possible to stop the rock at the photon sphere.

Now, if the rock is stopped at the photon sphere, it will once again have only its rest energy $E = mc^2$. So, the process of stopping the rock releases $\Delta E = mc^2(\sqrt{3} - 1)$. However, in order to turn the generator, this energy must be transported back up the string. Transporting the energy necessarily decreases it by the factor $\sqrt{1 - R_s/r} = \sqrt{\frac{1}{3}}$. So, the energy that is available far away to turn the generator is $\Delta E_\infty = 3^{-1/2}\Delta E = mc^2(1 - 1/\sqrt{3}) \sim (.42)mc^2$. According to the problem, all of this energy is then converted to electricity. To put some numbers in, for a $1kg$ rock (or $1kg$ of kitchen garbage....), this is $\Delta E_\infty \sim (.42)(1kg)(3 \ \times \ 10^8 m/s)^2 = 4 \ times \ 10^{16} J \sim 10^{10} kW - hrs$. That is, roughly enough electricity to power to the entire United States for one day. There's no waste left over either.