Solutions to Homework Assignment #10 – PHY312

9-1. Recall that the Schwarzschild metric is:

$$ds^2 = -(1 - R_s/r)dt^2 + \frac{dr^2}{1 - R_s/r} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$

A static clock does not move in r, θ, or ϕ. So, along its worldline we have $dr = 0$, $d\theta = 0$, and $d\phi = 0$. As a result, the proper time τ measured by this clock satisfies:

$$d\tau^2 = (1 - R_S/r)dt^2,$$

or, $d\tau = dt\sqrt{1 - R_s/r}$.

A) Since this does not depend on t, we easily integrate both sides to get $\Delta\tau = \Delta t\sqrt{1 - R_s/r}$.

Now, how fast does a clock run at $r = \infty$? If we take the limit of the above result as $r \to \infty$, we find $\Delta\tau = \Delta t$. So, for a clock at $r = \infty$, Δt actually represents the *proper time* measured by that clock. As a result, the proper time $\Delta\tau$ measured on a clock at a *finite* value of r is related to the proper time Δt measured by a clock 'at infinity' by $\Delta\tau = \Delta t\sqrt{1 - R_s/r}$.

B) For the sun, $R_S = 2M_{Sun}G/c^2 = 2(2 \times 10^{30} kg)(6.7 \times 10^{-11} Nm^2/kg^2)/(3 \times 10^8 m/s)^2 = 3 \ \times 10^3 m = 3km$. So, $R_S/r = 4 \ \times \ 10^{-6}$. Thus, the time measured by your clock is $(1 - 2 \ \times \ 10^{-6})$ times the time measured far away. (Your clock runs slower by 2 parts in a million, a difference of approximately 1 minute per year.)

C) Recall that the acceleration of a static clock relative to freely falling observers (in a time-independent gravitational field in 1+1 dimensions) is given by $\alpha = \frac{1}{2}\frac{d}{ds} \ln g_{tt}$, where the derivative with respect to proper distance s is taken along a line at constant t going upwards in the gravitational field. Thus, along this line $dt = d\theta = d\phi = 0$, and we have

$$ds = \frac{dr}{\sqrt{1 - R_s/r}}.$$

Thus,

$$\alpha = \frac{c^2}{2}\frac{dr}{ds}\frac{d}{dr} \ln g_{tt} = \frac{c^2}{2}\left(\frac{R_s}{r^2}\right)(1 - R_s/r)^{-1/2}.$$

D) At the surface of the sun, we saw in part B that $R_s/r = 5 \times 10^{-6}$. So, $1 - R_s/r$ is essentially just 1. Thus, $\alpha = (c^2/2)(4 \times 10^{-6})(7 \times 10^8 m)^{-1} = 2.5 \times 10^2 m/s^2 = 250 m/s^2 = 25g$. So, you would feel 25 times as heavy as you do here on the surface of the earth.

9-3. OK, now we want to compute the proper time measured by an *orbiting* clock. Here, I have given you the important fact that, in this metric, the clock orbits at a speed such that:

$$\frac{d\phi}{dt} = \sqrt{\frac{R_s}{2r^3}}.$$

A) For a clock in a *circular* orbit around the equator, $\theta = \pi/2$ is a constant, and r is a constant. Thus, $dr = d\theta = 0$. This time, however, neither $d\phi$ nor dt is zero. Instead, dt and $d\phi$ are related by the equation above: $d\phi = dt\sqrt{R_s/2r^3}$. Let us substitute this into the metric to express the proper time $d\tau$ in terms of dt:

$$d\tau^2 = -ds^2 = (1 - R_s/r)dt^2 - r^2 d\phi^2 = (1 - R_s/r)dt^2 - r^2 \frac{R_s}{2r^3} dt^2 = (1 - \frac{3R_s}{2r})dt^2.$$

In other words, $d\tau = (1 - \frac{3R_s}{r})dt$, and $\tau = (1 - \frac{3R_s}{2r})t$.

B) At $r = 3R_s/2$, we have $d\tau = 0$!!

C) Note that, for $r < 3R_s/2$, we have $d\tau^2 < 0$. But this means that the 'orbit' is *spacelike* (i.e., moving faster than the speed of light) rather than timelike. (Remember this from way back in the first part of the semester?) In other words, the orbit at $r = 3R_s/2$ is a lightlike worldline – the worldline of a ray of light! Thus, light rays will orbit at $r = 3R_s/2$. This means that below $r = 3R_s/2$, nothing can orbit along a circular path (since to do so it would be moving faster than light).

In fact, *any* freely-falling object around a black hole that ever dips below $r = 3R_s/2$ must fall through the horizon. To see this, let's assume the converse – i.e., that the object can fall below $r = 3R_s/2$ and then return. In this case, at some point in its orbit it reaches a point of minimum r, where it begins to move back outward. Since, at this point, $dr/dt = 0$, it is not moving in the r direction at all. Instead, it is moving only *around* the black hole. So, for that instant it is orbiting along a piece of a circle around the black hole. Now, since $r < 3R_s/2$, even a beam of light pointed along that circle would fall in closer to the black hole. Since our object cannot move faster than the speed of light, it too must fall in. Thus, this was not a point of minimum r after all.

9-5. A) So, is it possible for an observer to remain static at the photon sphere? Well, that's outside the horizon, so the answer should be 'yes.' In particular, remember that the photon sphere is at $r = 3R_s/2$. Also remember that we have a general formula for the proper acceleration of a static observer outside the horizon:

$$\alpha = \frac{c^2}{2}\left(\frac{R_s}{r^2}\right)(1 - R_s/r)^{-1/2}.$$

Setting $r = 3R_s/2$, we find that an observer at the photon sphere requires an acceleration of

$$\alpha = \frac{c^2}{2}\left(\frac{4}{9R_s}\right)(1 - 2/3)^{-1/2} = \frac{2c^2}{9R_s}\sqrt{3}.$$

So, if R_s is big enough, this will even be a small value.

B) If the black hole is big enough that the above acceleration is small, then you would be able to survive! Let's try some numbers. For a black hole of the same mass as the sun, $R_s = 3000m$. The corresponding acceleration is $1 \times 10^{13} m/s^2$ – way too big to survive.

But what about for a really big black hole? Suppose, for example, that R_s was one light year. Then, $c^2/R_s = 1(light - year)/yr^2 = 1g = 10m/s^2$. So, the acceleration of a static observer at the photon sphere would be $2\sqrt{3}/9$ times this: $4m/s^2$. This would be easy to survive – in fact, you would feel only 40% of your weight on earth. A black hole with a Schwarzschild radius of (1 light-year $= 10^{16}m$) is 3×10^{13} times bigger than a black hole with the mass of the sun, and so would have a mass 10^{13} times bigger than the sun (for a total of $2 \times 10^{43} kg$.)

Hmmm... why didn't I ask about tidal forces here? I mean, OK, so the above considerations are clearly relevant, but aren't there tidal forces as well? To understand this, remember that tidal forces arise from the difference between the acceleration your head requires to remain static and the corresponding number for your feet. If you are freely falling, this leads to a stretching as the various parts of your body 'fall at different rates' (relative to static observers). Here, you are not in fact falling! If the force on your feet is small enough not to damage them, how could the rest of your body be concerned by the fact that the corresponding forces holding up your other body parts are even smaller?