

Lecture 4

- Summary so far ...
- Nonlinear systems – chaos
- Phase space, Poincare maps, strange attractors
- Period doubling
- Lorenz model, balls in boxes ...

Summary so far

- Learnt how to solve Newton's laws of motion for simple mechanical systems – 1 or more particles in 1 or more dimensions.
- Euler or leapfrog algorithms.
- Example: molecular model of gas
- Could go on – eg use Newton's law of gravity to model solar system
- Motions are *deterministic* – nevertheless motion can appear complicated or even random (many particles ...)
- But it cannot be truly random ! In fact it is *chaotic*

Real pendulum

Variables $\theta(t)$, $\omega(t)$ equation of motion:

$$\begin{aligned}\frac{d\omega}{dt} &= -\frac{g}{l} \sin(\theta) - k \frac{d\theta}{dt} + F \sin(\omega_D t) \\ \frac{d\theta}{dt} &= \omega\end{aligned}$$

Integrate/solve by introducing discrete time $t = ndt, n = 1 \dots$ and find recurrence relations of form $\theta_{n+1} = \theta_n + \dots$

Exactly analagous to 1D motion with $x \rightarrow \theta$,
 $p \rightarrow \omega$

Use same code!

(note: θ angle - restrict to $-\pi \rightarrow \pi$)

Plotting graphs

- Additional displays to show 2D plots are easily created

```
pic=gdisplay()  
myplot=gcurve(gdisplay=pic,...)
```

- Data is plotted with the command `myplot.plot(pos=(x,y))`
- Can specify how big is display and where it appears on computer screen. Also labels, axes etc
- Multiple windows/displays allowed

Simple observations

- Initially transients seen - remnant of decaying natural oscillation
- Small driving force, small amplitude, motion in step with driving force - like *harmonic* case
- Larger F – apparently random or *chaotic* behavior seen.
- Windows of regular motion found at larger F !
- *Cannot* be truly random - motion deterministic. Something more subtle happening ...

Sensitivity to initial conditions

Two identical pendula with *slightly* different initial conditions.

- In regular regime: motions *converge* with time
- In chaotic regime : diverge!
- In first case poor knowledge of initial conditions is *irrelevant* to predicting long time motion
- In other case implies *no predictability* at long times (eg. weather ...)

Phase space

Useful to examine motion not as (t, θ) and (t, ω) but in *phase space* (θ, ω) .

- Regular (non-chaotic) motion yields *simple closed curve*.
- Chaotic motion – much structure. Many nearly closed orbits, sudden departures to new orbits, never repeating.

Poincare plots

Instead of plotting entire phase space trajectory, plot (θ, ω) *only* at multiples of time period of driving force.

- For regular motion - single point seen.
- For chaotic motion - non space filling structure seen. Does *not* depend on initial conditions
- Predictable aspect of chaotic motion – called a *strange attractor*.
- *All* chaotic motions of system approach a motion on the attractor.
- *Not* a 1D curve – in general *fractal* object - later.

Period doubling

- At $F = 1.35$ motion has same period as driving force
- At $F = 1.44$ we see motion has *twice* period of driving force
- At $F = 1.465$ four times driving period $T = T_D$
- Continues. Successively smaller increases in F yield doublings of the period of the motion. $T = \infty$ at finite F !
- Period doubling route to chaos common. Furthermore

$$\delta_n = \frac{F_n - F_{n-1}}{F_{n+1} - F_n} \lim_{n \rightarrow \infty} = \delta \sim 4.669..$$

Feigenbaum delta

Lorenz model

- Another example of model showing chaos.
- (Very)-simplified model of convective fluid flow – container containing fluid with bottom and top surfaces held at different temperatures.
- Three variables x, y, z corresponding to temperature, density and fluid velocity
- Three parameters σ, r, b (temperature difference and fluid parameters)
- Full solution involves *Navier-Stokes* and very many variables. Weather simulations etc.

Lorenz equations

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y - x) \\ \frac{dy}{dt} &= -xz + rx - y \\ \frac{dz}{dt} &= xy - bz\end{aligned}$$

Discretize time and solve as before

Set $\sigma = 10.0$, $b = 8/3$. r measures temperature difference. Analogous to F in pendulum example.

$r = 5$ - settles to point - simple convective flow.

$r = 25$ - chaos - Lorenz attractor - chaotic or turbulent flow